Enrollment No: ______________________ Exam Seat No: ______________________

C.U.SHAH UNIVERSITY

Summer Examination-2017

Subject Name: Advanced Algebra

Subject Code: 5SC04AAE1 **Branch: M.Sc.(Mathematics)**

Semester: 4 **Date: 15/04/2017** **Time: 10:30 To 01:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions.** **(07)**

a) The sequence $0 \to M \xrightarrow{f} N$ is exact if and only if f is ______. **(01)**

b) State first isomorphism theorem. **(01)**

c) True or False: The additive group $(Q, +)$ is finitely generated. **(01)**

d) Prove that composition of two module homomorphism is homomorphism. **(02)**

e) Let M be R module. Then prove that $0 \to M \to 0$ is exact if and only if $M = 0$. **(02)**

Q-2 **Attempt all questions** **(14)**

a) State and prove second isomorphism theorem. **(07)**

b) Prove that every internal direct sum of modules is also an external direct sum. **(05)**

c) Prove that an$R$-module M is simple if and only if it is cyclic and every non zero element is its generator. **(02)**

OR

Q-2 **Attempt all questions** **(14)**

a) State and prove correspondence theorem. **(07)**

b) Let M be a simple R module. Show that the $End(M)$ is a division ring. **(05)**

c) Show that the $\boldsymbol{Z}$module $\boldsymbol{Q}$is not free. **(02)**

Q-3 **Attempt all questions** **(14)**

a) Prove that a short exact sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ splits and $M \cong M_1 \oplus M_2$if and only if there exists $h: M \to M_1$ with $hf = 1_{M_1}$. **(05)**

b) Prove that if M is simple R module, then M is isomorphic to R/I for some maximal left ideal $I$of R. **(05)**

c) Let M, N be R module and $\varphi: M \to N$ be a module homomorphism then prove that the range of φ is submodule of N. **(04)**

OR

Q-3 **Attempt all Questions** **(14)**

a) If M is a module over a ring R, then prove that $HOM_R(R, M)$ is an R module with the action of R on $HOM_R(R, M)$ given by $(rf)(x) = f(xr)$, where $f \in HOM_R(R, M)$and $r, x \in R$. **(05)**

b) Prove that let R be a commutative ring with identity the mapping $M_n(R) \to M_n(R)$defined by $A \to A^t$is an R module isomorphism. **(05)**

c) Show that$< X >_s = < X >_I$, whenever R has identity. If we remove identity from R then above result is true? Justify. **(04)**

SECTION – II

Q-4 **Attempt the Following questions.** **(07)**

a) True or False: $2\boldsymbol{Z}$over $2\boldsymbol{Z}$module is free. **(01)**

b) True or False: Any cyclic module over principal ideal domain is simple. **(01)**

c) Define: Free module. **(01)**

d) Consider $\boldsymbol{Z}$as a $\boldsymbol{Z}$module then prove that $B = \{1\}$ is a basis for $\boldsymbol{Z}$. **(02)**

e) Prove that every commutative ring with identity has invariant rank property. **(02)**

Q-5 **Attempt all questions** **(14)**

a) If M is finitely generated module over principal ideal domain$R$ then show that $M = T(M) \oplus F$, where F is a free module of finite rank. **(07)**

b) Let D be division ring and M be a D module then prove that M is free D module. **(07)**

OR

Q-5 **Attempt all Questions** **(14)**

a) Let R be a ring and $M_1, M_2, M_3, \dots, M_n$ be R module. Then prove that $M \cong M_1 \oplus M_2 \oplus M_3 \oplus \dots \oplus M_n$if and only if for each $l \in N$there are R module homomorphisms$P_l: M \to M_l$ and $U_l: M_l \to M$ such that **(07)**

i) $P_l U_l = 1_{M_l}$

ii) $P_k U_l = 0$ for $k \neq l$

iii) $\sum_{l=1}^{n} U_l P_l = 1_M$

b) Define projective and injective module. Prove that a free module is projective. **(07)**

Q-6 **Attempt all questions** **(14)**

a) Let I be a proper ideal of a ring R and M be a free R module with a basis $B = \{x_\lambda : \lambda \in \Lambda\}$.Then prove that $\{x_\lambda + IM : \lambda \in \Lambda\}$ is a basis for $M|IM$. **(07)**

b) Define invariant rank property. Prove that if M is a module over a division ring D,then D has an invariant rank property. **(07)**

OR

Q-6 **Attempt all Questions** **(14)**

a) Prove that a finitely generated torsion free module over principal ideal domain is free. **(07)**

b) Prove that let M be a free R module and $B \subset M$. $B$is a basis of M if and only if every element of M can be uniquely written as linear combination of B. **(07)**